UNIQUENESS AND STABILITY OF THE SOLUTION OF THE SMALL PERTURBATION PROBLEM OF A FLEXIBLE FILAMENT WITH A FREE END

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An inextensible flexible filament having slight curvature and slight torsion of the shape in the unperturbed state is considered. The question of the uniqueness and stability of the solution of the small perturbations problem is investigated for the case when one end of the filament is free. A particular case of such a problem has been investigated in [1].

Let us start from the vector equation of dynamics of a filament

$$\rho \frac{\partial^2 \mathbf{r}}{\partial t^2} = \frac{\partial}{\partial s}\left(T \frac{\partial \mathbf{r}}{\partial s}\right) + \mathbf{F}, \qquad \left|\frac{\partial \mathbf{r}}{\partial s}\right| = 1 \tag{1}$$

Here ρ is the density, $\mathbf{r}$ the radius-vector of points of the filament, T the tension, $\mathbf{F}$ external forces, s the arc coordinate, and t the time.

There follows from (1) for small perturbations Δr and ΔT:

$$\rho \frac{\partial^2 \Delta \mathbf{r}}{\partial t^2} = \frac{\partial}{\partial s}\left(T_0 \frac{\partial \Delta \mathbf{r}}{\partial s}\right) + \frac{\partial}{\partial s}\left(\Delta T \frac{\partial \mathbf{r}_0}{\partial s}\right), \qquad \frac{\partial \mathbf{r}_0}{\partial s} \frac{\partial \Delta \mathbf{r}}{\partial s} = 0 \tag{2}$$

Here $T_0(s)$, $\mathbf{r}_0(s)$ are unperturbed values of T and $\mathbf{r}$. Let us expand the perturbation $\Delta\mathbf{r}$ in the natural axes $\boldsymbol{\tau}_0$, $\mathbf{n}_0$, $\mathbf{b}_0$, corresponding to the shape of the filament in the unperturbed state $\qquad \Delta \mathbf{r} = \Delta r_1 \boldsymbol{\tau}_0 + \Delta r_2 \mathbf{n}_0 + \Delta r_3 \mathbf{b}_0$

Let us first assume that the shape of the filament in the unperturbed state is similar to a plane, i.e. the quantities $\partial \mathbf{b}_0 / \partial s$, $\partial^2 \mathbf{b}_0 / \partial s^2$, $\partial^2 \boldsymbol{\tau}_0 / \partial s^2$, $\partial \mathbf{n}_0 / \partial s$ have the same order of smallness as the perturbations $\Delta \mathbf{r}$, ΔT. Then multiplying (2) scalarly by $\mathbf{b}_0$, we obtain

$$\rho \frac{\partial^2 \Delta r_3}{\partial t^2} = \frac{\partial}{\partial s}\left(T_0 \frac{\partial \Delta r_3}{\partial s}\right) \tag{3}$$

Therefore, the binormal small perturbations can be studied independently of the rest.

If the shape of the filament in the initial state has slight curvature, i.e. the quantities $\partial \boldsymbol{\tau}_0 / \partial s$ have the same order of smallness as the perturbations $\Delta \mathbf{r}$, ΔT, then multiplying (2) scalarly by $\mathbf{n}_0$ and $\boldsymbol{\tau}_0$, we obtain

$$\rho \frac{\partial^2 \Delta r_2}{\partial t^2} = \frac{\partial}{\partial s}\left(T_0 \frac{\partial \Delta r_2}{\partial s}\right), \qquad \rho \frac{\partial^2 \Delta r_1}{\partial t^2} = \frac{\partial \Delta T}{\partial s} \tag{4}$$

Consequently, all the components of the small perturbations can be studied independently in the last case.

Let us first examine Eq. (3) by assuming that $\rho(s)$ and $T_0(s)$ are known functions characterizing the unperturbed state of the filament. If the filament is fastened at two points ($s = 0$, $s = l$), or its motion at these points is given kinematically, then there exists a unique solution satisfying the initial and boundary conditions [2]

$$\Delta r_3(s, 0) = \varphi(s), \qquad [\partial \Delta r_3 / \partial t]_{t=0} = \psi(s) \tag{5}$$

$$\Delta r_3(0, t) = \mu_1(t), \qquad \Delta r_3(l, t) = \mu_2(t)$$

At the point $s = l$ let the filament have a free end

$$T_0 \,/_{s=l} = 0 \tag{6}$$

Then the second boundary condition in (5) falls away. Let us show that the uniqueness of the solution is assured even in this case, but under certain constraints.

Following [2], let us assume that there exist two solutions of the problem under consideration: $\Delta r_3' \,(s,\, t)$ and $\Delta r_3'' \,(s,\, t)$, and let us examine the difference

$$v\,(s,\, t) = \Delta r_3' \,(s,\, t) - \Delta r_3'' \,(s,\, t)$$

The function $v\,(s,\, t)$ satisfies the homogeneous equation with additional homogeneous conditions

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial s}\left(T_0 \frac{\partial v}{\partial s}\right), \quad v\,(s,\, 0) = 0, \quad \left[\frac{\partial v}{\partial t}\right]_{t=0} = 0, \quad v\,(0, t) = 0$$

Let us consider the energy

$$E\,(t) = \frac{1}{2}\int_0^l \left\{T_0\left(\frac{\partial v}{\partial s}\right)^2 + \rho\left(\frac{\partial v}{\partial t}\right)^2\right\} ds$$

Evidently

$$\frac{dE\,(t)}{dt} = \int_0^l \left(T_0 \frac{\partial v}{\partial s}\, \frac{\partial^2 v}{\partial s \partial t} + \rho\, \frac{\partial v}{\partial t}\, \frac{\partial^2 v}{\partial t^2}\right) ds$$

and

$$\int_0^l T_0\, \frac{\partial v}{\partial s}\, \frac{\partial^2 v}{\partial s \partial t} ds = \left(T_0 \frac{\partial v}{\partial s}\, \frac{\partial v}{\partial t}\right)\Big|_0^l - \int_0^l \frac{\partial v}{\partial t}\, \frac{\partial}{\partial s}\left(T_0 \frac{\partial v}{\partial s}\right) ds$$

The substitution vanishes because of the boundary condition $v\,(0,\, t) = 0$ and condition (6). Therefore

$$\frac{dE\,(t)}{dt} = \int_0^l \frac{\partial v}{\partial t}\left[\rho \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial s}\left(T_0\, \frac{\partial v}{\partial s}\right)\right] ds = 0, \qquad E\,(t) = \text{const}$$

Taking the initial conditions into account, we obtain

$$E\,(t) = E\,(0) = \frac{1}{2}\int_0^l \left\{T_0\left(\frac{\partial v}{\partial s}\right)^2 + \rho\left(\frac{\partial v}{\partial t}\right)^2\right\}_{t=0} ds = 0 \tag{7}$$

In contrast to [2], there still does not here follow from (7) that

$$v\,(s,\, t) = 0 \tag{8}$$

since weaker inequalities hold than in [2], namely:

$$\rho\,(s) > 0 \quad (0 \leqslant s \leqslant l), \;\; T_0\,(s) > 0 \;\; (0 \leqslant s < l), \;\; T_0 \,/_{s=l} = 0$$

And only if the solution is sought in the class of functions in which the derivative $\partial v \,/\, \partial s$ is continuous in a closed interval $0 \leqslant s \leqslant l$ does (8) result from (7), and consequently the solution is unique.

Such a singularity of the formulated problem is associated with the fact that the initial equation is hyperbolic in the open interval $0 \leqslant s < l$ but degenerates into a parabolic equation in the closed interval $0 \leqslant s \leqslant l$.

Let us investigate the stability of the solution in the sense of correctness in the formulation of the problem with initial conditions. Let

$$\varphi\,(s) \begin{cases} > 0 & \text{for} \quad 0 < s_2^\circ < s < s_1^\circ < l \\ = 0 & \text{for} \quad s_1^\circ \leqslant s \leqslant l \end{cases}$$

$$\psi\,(s)\,|_{0 \leqslant s \leqslant l} = 0, \qquad \mu_1\,(t) = 0, \qquad t \geqslant 0 \tag{9}$$

Then

$$E(t) = \frac{1}{2}\int_0^l \left[\rho\left(\frac{\partial \Delta r_3}{\partial t}\right)^2 + T_0\left(\frac{\partial \Delta r_3}{\partial s}\right)^2\right] ds =$$

$$= \frac{1}{2}\int_{s_1}^{s_2} \left[\rho\left(\frac{\partial \Delta r_3}{\partial t}\right)^2 + T_0\left(\frac{\partial \Delta r_3}{\partial s}\right)^2\right] ds = E_0 = \mathrm{const} > 0 \tag{10}$$

since

$$\frac{dE(t)}{dt} = T_0 \frac{\partial \Delta r_3}{\partial s}\frac{\partial \Delta r_3}{\partial t}\Bigg|_0^l - \int_0^l \frac{\partial \Delta r_3}{\partial t}\left[\rho\frac{\partial^2 \Delta r_3}{\partial t^2} - \frac{\partial}{\partial s}\left(T_0 \frac{\partial \Delta r_3}{\partial s}\right)\right] ds = 0$$

Here s_1 and s_2 are arclength coordinates of the leading and trailing fronts of the discontinuity wave of the derivatives $\partial^2 \Delta r_3 / \partial t^2$ and $\partial^2 \Delta r_3 / \partial s^2$, where $s_1 = s_1^\circ$, $s_2 = s_2^\circ$ at $t = 0$.

Evidently

$$\Delta r_3 \equiv 0 \qquad (s < s_1,\ s > s_2)$$

From the differential equations of the characteristics

$$ds_1 / dt = \pm (T_0 / \rho)^{1/2}, \quad ds_2 / dt = \pm (T_0 / \rho)^{1/2}$$

we find the equations of the characteristics passing through s_1° and s_2°

$$t = \int_{s_1^\circ}^{s_1} \frac{d\zeta}{\sqrt{T_0(\zeta)/\rho(\zeta)}}, \quad t = \int_{s_2^\circ}^{s_2} \frac{d\zeta}{\sqrt{T_0(\zeta)/\rho(\zeta)}} \quad (0 < s_1, s_2 < l)$$

A singular solution coincident for both characteristics holds for $s_{1,2} = l$ because of (6). Two cases may arise:

1°. The improper integral

$$\int_{\zeta_0}^{\zeta} \frac{d\zeta}{\sqrt{T_0(\zeta)/\rho(\zeta)}} \tag{11}$$

converges for $\zeta \to l$, i.e. coincidence of the characteristics occurs for finite $t = t^*$. Then

$$s_2 - s_1 \to 0 \quad \text{for } t \to t^*$$

But, as follows from (10), in this case

$$\rho\left(\frac{\partial \Delta r_3}{\partial t}\right)^2 + T_0\left(\frac{\partial \Delta r_3}{\partial s}\right)^2 \to \infty \quad \text{for } s \to l$$

and because of the boundedness of ρ and T_0, taking account of (6), we arrive at the estimate

$$\partial \Delta r_3 / \partial t \to \infty \qquad \text{for} \quad s \to l \tag{12}$$

which holds for arbitrarily small initial conditions $\varphi(s)$.

The estimate (12) indicates incorrectness in the formulation of the problem in the closed interval $0 \leqslant s \leqslant l$, i.e. instability of the solution near the free end.

The result formulated has been obtained under the particular initial conditions (9). However, by using the superposition principle for the original linear equation, and adding the conditions (9), which may differ from the zero conditions as little as desired, to the arbitrary initial conditions, we arrive at the same estimate (12).

The problem considered has been obtained as a result of linearizing the original physical problem, hence, the mathematical formulation of the instability should be weakened as compared with the estimate (12) and is written in the form

$$\partial \Delta r_3/\partial t \geqslant \delta > 0 \quad \text{for} \quad \varphi(s) \to 0$$

A sharp rise in the velocities of points of the filament near the free end (the crack of a whip) is the physical manifestation of the instability remarked above. The convergence of the integral (11) occurs in the majority of cases of practical importance, and particularly when a homogeneous weighted filament has a straight-line unperturbed shape.

2°. Let the improper integral (11) diverge as $\eta \to l$, i.e. the characteristics coincide as $t^* \to \infty$. Then the estimate (12) does not hold, where any perturbation originating in the interval $0 \leqslant s < l$ will not reach the free end in a finite time interval, and any perturbation originating at the free end will not be propagated to the remaining points of the filament. In other words, the free end becomes an isolated point at which the value of the function is not at all connected with the values of the function at the remaining points.

This circumstance is indeed an illustration of that ambiguity in the solution which has been remarked in the investigation of (7). Indeed, uniqueness holds here in the open interval $0 \leqslant s < l$ but is violated in the closed interval $0 \leqslant s \leqslant l$.

The results presented refer to small binormal perturbations described by Eq. (3). If the filament has slight curvature in the unperturbed state, then the small normal perturbations are described by (4), which agrees completely with (3). Therefore, all the results obtained relative to small binormal perturbations, go over completely into small normal perturbations in this case.

The results obtained can be extended to the case when the filament moves in a resistive medium, i.e. when the external load $\mathbf{F}$ is a follower force. Let us set in (1)

$$\mathbf{F} = F_1\boldsymbol{\tau}_0 + F_2\mathbf{n}_0 + F_3\mathbf{b}_0$$

and let

$$F_i = \Phi_i\left(\frac{\partial r_i}{\partial t}\right), \qquad \Delta F_i = -\mu_i \frac{\partial \Delta r_i}{\partial t}, \qquad \mu_i = -\frac{\partial \Phi_i}{\partial(\partial r_i/\partial t)}$$

Here μ_i is the coefficient of resistivity of the medium. Then Eq. (3) becomes

$$\rho\frac{\partial^2 \Delta r_3}{\partial t^2} = \frac{\partial}{\partial s}\left(T_0\frac{\partial \Delta r_3}{\partial s}\right) - \mu_3\frac{\partial \Delta r_3}{\partial t}$$

where

$$\frac{dE(t)}{dt} = -\int_{s_1}^{s_2}\mu_3\left(\frac{\partial \Delta r_3}{\partial t}\right)^2 ds, \quad E = \frac{1}{2}\int_{s_1}^{s_2}\left[\rho\left(\frac{\partial \Delta r_3}{\partial t}\right)^2 + T_0\left(\frac{\partial \Delta r_3}{\partial s}\right)^2\right]ds \qquad (13)$$

From (13) there follows

$$E = E_0 - \int_{s_1}^{s_2}\int_0^t \mu_3\left(\frac{\partial \Delta r_3}{\partial t}\right)^2 dt\,ds$$

that is

$$\int_{s_1}^{s_2}\left[\rho\left(\frac{\partial \Delta r_3}{\partial t}\right)^2 + T_0\left(\frac{\partial \Delta r_3}{\partial s}\right)^2 + \mu_3\int_0^t\left(\frac{\partial \Delta r_3}{\partial t}\right)^2 dt\right]ds = E_0 = \text{const} \qquad (14)$$

If the improper integral in (11) converges, then $t \to t^*$ as $s_1 - s_2 \to 0$, where t^* is a finite number.

Hence, the estimate (12) follows from the unbounded growth of the integrand in (14) as $s \to l$. Therefore, in a resistive medium the effect of an abrupt rise in the velocities at the free end of a filament occurs under the same conditions as in a vacuum. An analogous result can undestandably be obtained also for Eq. (4) in the presence of resistive forces.

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ON THE APPLICATION OF NONSTATIONARY ANALOGY FOR THE DETERMINATION OF HYPERSONIC FLOWS PAST BLUNT BODIES

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A class of nonstationary flows for which the nonstationary analogy may be directly used for calculating the hypersonic flow past a blunt body at some distance from its blunted part is indicated.

It is shown that the nonstationary analogy without the introduction in the flow field of entropy corrections, generally, necessitates a certain special distortion of the shock wave shape during the transition from a nonstationary to a stationary flow.

The known solutions of equations of the nonstationary motion of gas, and the similarity of the stationary and nonstationary flows, established in [1–3], are often used in calculations of hypersonic flows around blunted slender bodies. The most frequently used are the solutions for strong explosions [4] and those for a piston moving according to the power law [5]. However, with the latter method of constructing solutions the entropy at the surface of the body corresponding to the piston is infinite, and in the neighborhood of this surface the solution looses its physical meaning.

This shortcoming of the theory has been corrected in [6–10] in which the inverse problem, i.e. that of finding a stationary flow corresponding to a shock, and obtained from the initial nonstationary solution by the nonstationary analogy (by the substitution $x = u_\infty t$). The method of introducing the so-called entropy corrections to the shape of the body and to flow parameters derived directly from nonstationary analogy was proposed in those papers. It was shown in [9] that, in particular, in the case of a strong explosion such corrections reduce to a modification of only the shape of the body, whose surface must be assumed to follow the streamline corresponding to the trajectory of that particle of the nonstationary flow whose entropy equals that obtaining downstream of a normal shock. The complete class of flows having this property will be indicated in the following. The analysis of other flows in this formulation of the inverse problem is made much more difficult by the necessity of introducing corrections not only to the shape of the body but, also, to the flow field [10]. With the aim of determining a certain class of stationary flows around blunt bodies, in this paper we propose a different method of construction (of solutions) which avoids these difficulties. The underlying idea of this method is to introduce corrections to the shape of the bow shock derived by nonstationary analogy, and not to the flow field. The shape of this shock is selected on the basis of the condition of complete congruence of the fields of stationary and nonstationary